Journal of Statistical Physics, Vol. 47, Nos. 1/2, 1987

A Rigorous Control of Logarithmic Corrections in Four-Dimensional φ^4 Spin Systems. II. Critical Behavior of Susceptibility and Correlation Length

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Received November 17, 1986

Continuing the analysis started in Part I of this work, we investigate critical phenomena in weakly coupled φ^4 spin systems in four dimensions. Concerning the critical behavior of the susceptibility and the correlation length (in the high-temperature phase), the existence of logarithmic corrections to their mean field type behavior is rigorously shown (i.e., we prove $\chi(t) \sim t^{-1} |\ln t|^{1/3}$, $\xi(t) \sim t^{-1/2} |\ln t|^{1/6}$).

KEY WORDS: φ_4^4 spin systems; critical phenomena; logarithmic corrections; rigorous renormalization group.

1. INTRODUCTION AND MAIN RESULTS

In this paper, we continue the program started in the Part I of this work[(1)] (hereafter referred to as I), and extract the so-called *logarithmic corrections*[(2)] to the mean field predictions in the critical behavior of the susceptibility and correlation length of the weakly coupled φ_4^4 system.

The *Gibbs measure* of the φ^4 spin system on a d-dimensional hypercubic lattice $\Lambda_0 \subset \mathbf{Z}^d$ studied in this paper is defined as

$$d\mu(\Phi) = Z^{-1} \exp[-\mathscr{H}^0_{\Lambda_0}(\Phi)] \prod_x d\varphi_x \tag{1.1}$$

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0022-4715/87/0400-0099$05.00/0

where $x \in \Lambda_0 \subset \mathbf{Z}^d$, $\varphi_x \in \mathbf{R}$, $\Phi \equiv \{\varphi_x\}_x$,

$$\mathscr{H}^0_{\Lambda_0} \equiv \frac{1}{4} \sum_{|x-y|=1} (\varphi_x - \varphi_y)^2 + \sum_x \left[\frac{\mu_0}{2} \varphi_x^2 + \frac{\lambda_0}{4!} \varphi_x^4 \right] \tag{1.2}$$

and Z is determined by the normalization condition $\int d\mu(\Phi) = 1$. (By definition, μ_0 here differs from that of Part I by $\lambda_0 G_{0xx}/2 = \lambda_0 (G_0)_{00}/2$.) For any function F of Φ, its *thermal expectation value* is defined by

$$\langle F \rangle \equiv \int F \, d\mu(\Phi) \tag{1.3}$$

We take Λ_0 as a d-dimensional torus of side L^N. (We will mainly consider the system in four dimensions, i.e., $d=4$.)

As in I, we fix initial λ_0 to a sufficiently small (but positive) value, and vary only μ_0. We are interested in the behavior of (the infinite-volume limit of) the system, when μ_0 approaches its critical value μ_c [for the definition of μ_c, see (2.10)]. Our main physical result is the following:

Theorem 1.1. Consider the infinite-volume limit ($\Lambda_0 \to \mathbf{Z}^4$) of the four-dimensional φ^4 system (1.1), and vary μ_0, fixing $0 \leqslant \lambda_0$ ($\approx n_0^{-1} \leqslant \bar{n}_0^{-1} \ll 1$). Then:

(0) There exists a critical value $\mu_c(\lambda_0)$.

We write $t \equiv \mu_0 - \mu_c$.

(i) For $t > 0$, the *susceptibility* χ

$$\chi(t) \equiv \sum_{x \in \mathbf{Z}^4} \langle \varphi_0 \varphi_x \rangle \tag{1.4}$$

the *correlation length* ξ

$$\xi(t)^{-1} \equiv - \lim_{x_1 \to \infty} \frac{\ln \langle \varphi_0 \varphi_{(x_1,0,0,0)} \rangle}{x_1} \tag{1.5}$$

and the *renormalized coupling* g_{ren} (here $d=4$)

$$g_{\text{ren}}(t) \equiv \frac{-\bar{u}_4}{\chi^2 \xi^d} \tag{1.6}$$

$$\bar{u}_4 \equiv \sum_{x,y,z \in \mathbf{Z}^d} (\langle \varphi_0 \varphi_x \varphi_y \varphi_z \rangle - \langle \varphi_0 \varphi_x \rangle \langle \varphi_y \varphi_z \rangle - \langle \varphi_0 \varphi_y \rangle \langle \varphi_x \varphi_z \rangle - \langle \varphi_0 \varphi_z \rangle \langle \varphi_x \varphi_y \rangle) \tag{1.6'}$$

exist.

(ii) For $0<t$ sufficiently small, they satisfy the bounds:

$$R_1'(L, N_0, n_0)^{-2}\, t^{-1}\, |\ln t|^{1/3} \leqslant \chi(t) \leqslant R_1'^2 t^{-1}\, |\ln t|^{1/3} \tag{1.7}$$

$$R_1'(L, N_0, n_0)^{-1}\, t^{-1/2}\, |\ln t|^{1/6} \leqslant \xi(t) \leqslant R_1'\, t^{-1/2}\, |\ln t|^{1/6} \tag{1.8}$$

$$0 \leqslant g_{\text{ren}}(t) \leqslant C_g(L, N_0)(n_0 + |\ln t|)^{-1/2} \tag{1.9}$$

Here $L \geqslant 513$, $N_0 \geqslant 8$, $n_0 \sim \lambda_0^{-1}$ are the same as in I,

$$R_1'(L, N_0, n_0) \equiv C'(L, N_0)(n_0)^{C''(L, N_0)} \tag{1.10}$$

and C_g, C', and C'' are finite, positive constants depending only on L and N_0.

Remarks. 1. The theorem establishes the existence of the *logarithmic corrections* to the mean-field-like behavior in our weakly coupled φ_4^4 model.

2. For some general properties of the critical point (especially concerning its uniqueness), see Ref. 3.

3. For the implications of the logarithmic corrections (for all $\lambda > 0$) to the triviality problem of φ_4^4, see Ref. 4.

4. The n_0 ($\approx \lambda_0^{-1}$) dependence of the coefficient R_1' is of some interest. The fact that it diverges when $\lambda_0 \to 0$ ($n_0 \to \infty$) is quite reasonable from physical considerations, because in $\lambda_0 = 0$ (=Gaussian!!) theory, there are, of course, no logarithmic corrections.

5. The bound on g_{ren} can be improved[4]:

$$\begin{aligned} &0 \leqslant g_{\text{ren}}(t) \leqslant C_g(L, N_0; \delta)(n_0 + |\ln t|)^{-1+\delta} \\ &0 < \delta < 1 \qquad [C_g(L, N_0; \delta) \to \infty \text{ as } \delta \to 0] \end{aligned} \tag{1.11}$$

Note that, contrary to R_1', $C_g(L, N_0; \delta)$ does not depend on n_0.

By similar (but much simpler) analysis, we can prove:

Theorem 1.1′. Consider the same situation as in Theorem 1.1, but now for the φ^4 system in d dimensions ($d > 4$). Then we have:

(0), (i) The same as Theorem 1.1 (replace $\mathbf{Z}^4$ by $\mathbf{Z}^d$).

[4] We can further improve[(15)]:

$$g_{\text{ren}}(t) \approx (n_0 + |\ln t|)^{-1} + O[(n_0 + |\ln t|)^{-5/4}]$$

(both upper and lower bounds!). We are grateful to J. Chayes, L. Chayes, M. Aizenman, and A. Sokal for suggesting that improved bounds be obtained on g_{ren}.

(ii) For $0 < t$ sufficiently small, $\chi(t)$ and $\xi(t)$ satisfy the bounds:

$$\text{const} \cdot t^{-1} \leqslant \chi(t) \leqslant \text{const}' \cdot t^{-1} \tag{1.12}$$

$$\text{const} \cdot t^{-1/2} \leqslant \xi(t) \leqslant \text{const}' \cdot t^{-1/2} \tag{1.13}$$

We will prove Theorem 1.1 in Sections 2–4, following faithfully the original idea of Wilson,[5] but with complete mathematical rigor. First, in Section 2, we prove the theorem, making use of some bounds on the expectations $\langle \cdots \rangle_{\mathscr{H}^n}$, which are defined in terms of the nth effective Hamiltonian $\mathscr{H}^n$ [the choice of n is defined in (2.2)]. Then, in Sections 3–5, we prove these bounds on $\langle \cdots \rangle_{\mathscr{H}^n}$.

As for the continuum limit, we have the following result.

Proposition 1.2. The continuum limit of the theory

$$\mathscr{H}^0_{\Lambda_0} \equiv \frac{J}{4} \sum_{|x-y|=1} (\varphi_x - \varphi_y)^2 + \sum_x \left[\frac{\mu_0}{2} \varphi_x^2 + \frac{\lambda_0}{4!} \varphi_x^4 \right] \qquad (J \geqslant 0) \tag{1.14}$$

under the conditions

$$\mu_0 \geqslant \mu_c(\lambda_0, z) \qquad \text{and} \qquad 0 \leqslant \lambda_0 J^{-2} \leqslant (\bar{n}_0)^{-1} \tag{1.15}$$

is always *Gaussian* or *badly normalized* (i.e., all the correlation functions diverge, or all the correlation functions are zero!) theory.

Remark. The above proposition tells us that under the condition (1.15) the continuum limit is Gaussian (or meaningless), no matter what wave function renormalization we choose.

This proposition is proved by the bound on the *renormalized coupling* (Theorem 1.1) and/or by *direct analysis of correlation functions.* Details will be presented elsewhere (after the completion of the analysis of the low-temperature phase).

2. DERIVATION OF THE LOGARITHMIC CORRECTIONS

Before describing the detailed proofs, let us fix some notations. Note that we have to consider both the infinite-volume limit $\Lambda_0 \to \mathbf{Z}^d$ and the limit $\mu_0 \to \mu_c + 0$. Therefore, the definition of μ_c needs somewhat careful treatment. In the following, we will fix the initial value λ_0 and denote it as λ.

Definitions 2.1 (cf. I, Corollary 2.4). Let the original lattice Λ_0 be a four-dimensional torus of side L^N, with N sufficiently large. Define $\mu_c(\Lambda_0) = \mu_c(\lambda; \Lambda_0) =$ *critical value on* Λ_0 as in I, Theorem 2.1. (Of course, we need to take into account the difference between definitions of μ_0 in Part I and in Part II.) Now denote $t \equiv \mu - \mu_c(\Lambda_0) > 0$, and let

$$M(L, N_0) \equiv \alpha^{-2} L^{8N_0} \exp(55 + 8\alpha L^{N_0}) \tag{2.1}$$

Define $n_2 \geqslant 0$ as the smallest integer such that $\tilde{\mu}_{n_1+n_2} \geqslant M$. (For the definition of n_1, see I, Theorem 2.2.) In the following, we denote

$$n \equiv n_1 + n_2 \tag{2.2}$$

and also abbreviate $\mathscr{G} \equiv \mathscr{G}_n^{(\tilde{\mu}_n)}$, etc.

Remark. Note that the above M is still much smaller than our $\bar{n}_0$,

$$\bar{n}_0 \sim \exp(\alpha L^{N_0+1}) \tag{2.3}$$

We will treat several kinds of expectations, listed in the following.

Definitions 2.2.

$$\langle \cdots \rangle_{0,\Lambda,\mu} \equiv \frac{\int d\Phi \, \{\exp[-\mathscr{H}^0(\Phi)]\}(\cdots)}{\int d\Phi \exp[-\mathscr{H}^0(\Phi)]} \tag{2.4}$$

with $\mathscr{H}^0 = \mathscr{H}^0_{\Lambda_0}$ of (1.2), with $\mu_0 = \mu$ and $\Lambda_0 = \Lambda$.

$\langle \cdots \rangle_{n,\Lambda,\mu}$ is the expectation obtained by applying n BSTs to $\langle \cdots \rangle_{0,\Lambda,\mu}$, i.e., for a function F of Φ^n,

$$\langle F(\Phi^n) \rangle_{n,\Lambda,\mu} \equiv \langle F(z_n^{-1/2} \hat{C}^n \Phi) \rangle_{0,\Lambda,\mu} \equiv \frac{\int d\Phi^n \{\exp[-\mathscr{H}^n(\Phi^n)]\}(\cdots)}{\int d\Phi^n \exp[-\mathscr{H}^n(\Phi^n)]} \tag{2.4'}$$

We consider the torus Λ (considered as a set of sites and bonds) as a union of a *hypercube* and its *boundary* (a set of bonds that make the hypercube into a torus) $\partial\Lambda$. As for the infinite-volume limit of $\langle \cdots \rangle_{n,\Lambda,\mu}$,

$$\langle \cdots \rangle_{0,\mathbf{Z}^4,\mu} \equiv \lim_{\Lambda^f \to \mathbf{Z}^4} \langle \cdots \rangle^f_{0,\Lambda^f,\mu}$$
$$\langle \cdots \rangle^f_{0,\Lambda^f,\mu} \equiv \frac{\int d\Phi \{\exp[-\mathscr{H}^{0(\text{free})}_{\Lambda^f}(\Phi)]\}(\cdots)}{\int d\Phi \exp[-\mathscr{H}^{0(\text{free})}_{\Lambda^f}(\Phi)]} \tag{2.5}$$

and

$$\langle F(\Phi^n) \rangle_{n,\mathbf{Z}^4,\mu} \equiv \langle F(z_n^{-1/2} \hat{C}^n \Phi) \rangle_{0,\mathbf{Z}^4,\mu} \tag{2.5'}$$

where $\mathscr{H}^{0(\text{free})}_{\Lambda^f}$ is a Hamiltonian with free boundary condition, obtained from $\mathscr{H}^0_\Lambda$ (on Λ) by omitting the term $\sum_{|x-y|=1} \varphi_x \varphi_y$, where $\langle xy \rangle$ is contained in $\partial\Lambda$.

2.1. Bounds on Finite-Volume Expectations

For the expectations $\langle \cdots \rangle_{n,\Lambda,\mu}$ defined by the above $\mathcal{H}^n_\Lambda$ [with n of Eq. (2.2)], we can prove the following bounds (uniform in $|\Lambda_0|$).

Proposition 2.3. For $0 < t < (n_0)^{-2}$,

$$\langle \varphi_0^n \varphi_x^n \rangle_{n,\Lambda_0,\mu_c(\Lambda_0)+t} \leqslant \mathrm{const}(L, N_0) \exp(-\tfrac{1}{12}\alpha\,|x|) \tag{2.6}$$

$$(3L^2M)^{-1} \leqslant \chi_n \leqslant L^{4N_0+2} \tag{2.7}$$

where

$$\chi_n \equiv \sum_{x_n \in \Lambda_n} \langle \varphi_0^n \varphi_{x_n}^n \rangle_{n,\Lambda_0,\mu_c(\Lambda_0)+t} \tag{2.7$'$}$$

and

$$0 \leqslant -\bar{u}_{4,n} \leqslant \tfrac{1}{8} L^{16N_0+4}(n_0+n)^{-1/2} \tag{2.8}$$

where

$$\bar{u}_{4,n} \equiv \sum_{x,y,z \in \Lambda_n} \langle \varphi_0^n; \varphi_x^n; \varphi_y^n; \varphi_z^n \rangle \tag{2.8$'$}$$

Also abbreviating $1 \equiv (1, 0, 0, 0)$, $2 \equiv (2, 0, 0, 0)$, we have

$$(500L^2M)^{-1} \exp[-(2L^2M)^{1/2}] \leqslant \frac{\langle \varphi_0^n \varphi_2^n \rangle_{n,\Lambda_0,\mu_c(\Lambda_0)+t}}{\langle \varphi_0^n \varphi_1^n \rangle_{n,\Lambda_0,\mu_c(\Lambda_0)+t}} \tag{2.9}$$

holds for $|\ln t| \geqslant \mathrm{const}(L, N_0, M)$, sufficiently large.

This proposition is proven in Sections 3–5.

2.2. Bounds on Infinite-Volume Expectations

Note that, for the following two reasons, the two expectations $\langle \cdots \rangle_{n,\Lambda_0,\mu_c(\Lambda_0)+t}$ and $\langle \cdots \rangle_{n,\mathbf{Z}^4,\mu_c+t}$ may be different: (i) $\langle \cdots \rangle_{n,\Lambda_0,\mu_c+t}$ and its infinite-volume limit $\langle \cdots \rangle_{n,\mathbf{Z}^4,\mu_c+t}$ generally do not coincide. (ii) $\mu_c(\Lambda_0)$ may be different from μ_c (see below).

Concerning the $\Lambda_0 \to \mathbf{Z}^4$ limit of $\mu_c(\Lambda_0)$, we have (I, Theorem 2.5)

$$\lim_{\Lambda_0 \to \mathbf{Z}^4} \mu_c(\Lambda_0) = \mu_c \tag{2.10}$$

That is, the existence domain of $\mu_c(\Lambda_0)$ shrinks to a unique point of $\mathbf{R}$ as $\Lambda_0 \to \mathbf{Z}^4$.

For the $\Lambda_0 \to \mathbf{Z}^4$ limit of $\langle \cdots \rangle_{n,\Lambda_0,\mu_c+t}$, we have the following lemma.

Lemma 2.4. For fixed $t>0$, n, and for fixed x, there exists $\bar{N}(t)$ such that for $|\Lambda| = L^{4N}$, $N \geqslant \bar{N}$,

$$\langle \varphi_0 \varphi_x \rangle_{0,\Lambda,\mu_c+t} \leqslant \text{const}(t) \cdot e^{-m(t)|x|} \tag{2.11}$$

with $m(t)>0$, $\text{const}(t)<\infty$. Moreover, for fixed $t>0$ and $x \in \mathbf{Z}^4$,

$$\lim_{\Lambda \to \mathbf{Z}^4} \langle \varphi_0 \varphi_x \rangle_{0,\Lambda,\mu_c+t} = \langle \varphi_0 \varphi_x \rangle_{0,\mathbf{Z}^4,\mu_c+t} \tag{2.12a}$$

$$\lim_{\Lambda \to \mathbf{Z}^4} \langle \varphi_0 \varphi_x \rangle_{n,\Lambda,\mu_c+t} = \langle \varphi_0 \varphi_x \rangle_{n,\mathbf{Z}^4,\mu_c+t} \tag{2.12b}$$

Proof. Because $\mu_c(\Lambda) \to \mu_c$ as $\Lambda \to \mathbf{Z}^4$, we can take $\bar{N}(t)$ sufficiently large so that for $N \geqslant \bar{N}$, $\mu_c(\Lambda) + \frac{1}{2}t \leqslant \mu_c \leqslant \mu_c(\Lambda) + \frac{3}{2}t$. Then, by Proposition 2.3,

$$\langle \varphi_0^n \varphi_x^n \rangle_{n,\Lambda,\mu_c+t} \leqslant \text{const} \cdot e^{-(\alpha/12)|x|}$$

with $n \approx |\ln t|$. This in turn implies

$$\langle \varphi_0 \varphi_x \rangle_{0,\Lambda,\mu_c+t} \leqslant \text{const}' \cdot e^{-(\alpha/12)L^{-n}|x|} \tag{2.13}$$

Now the difference between $\langle \varphi_0 \varphi_x \rangle_{n,\Lambda,\mu_c+t}$ and $\langle \varphi_0 \varphi_x \rangle^f_{n,\Lambda^f,\mu_c+t}$ is

$$\begin{aligned}
&|\langle \varphi_0 \varphi_x \rangle_{n,\Lambda,\mu_c+t} - \langle \varphi_0 \varphi_x \rangle^f_{n,\Lambda^f,\mu_c+t}| \\
&= \left| \int_0^1 d\alpha \frac{\partial}{\partial \alpha} \langle \varphi_0 \varphi_x \rangle^\alpha_{n,\Lambda,\mu_c+t} \right| \\
&= \left| \int_0^1 d\alpha \sum_{|y-z|=1, \langle y,z \rangle \in \partial\Lambda} \langle \varphi_0 \varphi_x ; \varphi_y \varphi_z \rangle^\alpha_{n,\Lambda,\mu_c+t} \right|
\end{aligned} \tag{2.14a}$$

Here $\langle \cdots \rangle^\alpha$ is the expectation defined by the Hamiltonian $\mathscr{H}^{0,\alpha}$, where $\mathscr{H}^{0,\alpha}$ is defined by adding

$$-\frac{\alpha}{2} \sum_{|y-z|=1, \langle y,z \rangle \in \partial\Lambda} \varphi_y \varphi_z$$

to $\mathscr{H}^{0,f}$. By the Lebowitz[7] and Griffiths II[8] inequalities,

$$\begin{aligned}
0 &\leqslant \sum_{|y-z|=1, \langle y,z \rangle \in \partial\Lambda} \langle \varphi_0 \varphi_x ; \varphi_y \varphi_z \rangle^\alpha_{n,\Lambda,\mu_c+t} \\
&\leqslant 2 \sum \langle \varphi_0 \varphi_x \rangle^\alpha_n \langle \varphi_y \varphi_z \rangle^\alpha_n \\
&\leqslant 2 \sum \langle \varphi_0 \varphi_x \rangle_n \langle \varphi_y \varphi_z \rangle_n \\
&\leqslant \text{const}'(t) \exp[-\tfrac{1}{2} m(t) \min\{\text{dist}(x, \partial\Lambda), \text{dist}(y, \partial\Lambda)\}]
\end{aligned} \tag{2.14b}$$

On the other hand, the difference between $\langle \varphi_0 \varphi_x \rangle^{f}_{n,\Lambda^f,\mu_c+t}$ and $\langle \varphi_0 \varphi_x \rangle_{0,\mathbf{Z}^4,\mu_c+t}$ goes to zero, because the former converges to the latter by the monotonicity. This, together with the estimate (2.14b), leads us to the conclusion (2.12a). Equation (2.12b) immediately follows from the definition of $\langle \cdots \rangle_n$. ∎

Combining (2.10) and (2.12), we can finally obtain:

Proposition 2.5. Fix $t > 0$. Then, for fixed $x \in \mathbf{Z}^4$ ($|x| < \infty$)

$$\lim_{\Lambda \to \mathbf{Z}^4} \langle \varphi_0 \varphi_x \rangle_{0,\Lambda,\mu_c(\Lambda)+t} = \langle \varphi_0 \varphi_x \rangle_{0,\mathbf{Z}^4,\mu_c+t} \tag{2.15a}$$

and for fixed $m < \infty$,

$$\lim_{\Lambda \to \mathbf{Z}^4} \langle \varphi_0^m \varphi_x^m \rangle_{m,\Lambda,\mu_c(\Lambda)+t} = \langle \varphi_0^m \varphi_x^m \rangle_{m,\mathbf{Z}^4,\mu_c+t} \tag{2.15b}$$

Proof of Proposition 2.5. As was noted at the beginning of this section, the expectation $\langle \cdots \rangle_{n,\Lambda}$ may differ from $\langle \cdots \rangle_{n,\mathbf{Z}^4}$ for two reasons. We express them as

$$\begin{aligned} &|\langle \varphi_0 \varphi_x \rangle_{0,\Lambda,\mu_c(\Lambda)+t} - \langle \varphi_0 \varphi_x \rangle_{0,\mathbf{Z}^4,\mu_c+t}| \\ &\quad \leqslant |\langle \varphi_0 \varphi_x \rangle_{0,\Lambda,\mu_c(\Lambda)+t} - \langle \varphi_0 \varphi_x \rangle_{0,\Lambda,\mu_c+t}| \\ &\qquad + |\langle \varphi_0 \varphi_x \rangle_{0,\Lambda,\mu_c+t} - \langle \varphi_0 \varphi_x \rangle_{0,\mathbf{Z}^4,\mu_c+t}| \end{aligned} \tag{2.16}$$

We use the Lebowitz inequality[7] to bound the first term:

$$\begin{aligned} &|\langle \varphi_0 \varphi_x \rangle_{0,\Lambda,\mu_c(\Lambda)+t} - \langle \varphi_0 \varphi_x \rangle_{0,\mathbf{Z}^4,\mu_c+t}| \\ &\quad = \left| \int_{\mu_c+t}^{\mu_c(\Lambda)+t} d\mu \left\langle \varphi_0 \varphi_x ; -\frac{1}{2} \sum_y \varphi_y^2 \right\rangle_{0,\Lambda,\mu} \right| \\ &\quad \leqslant \left| \int_{\mu_c+t}^{\mu_c(\Lambda)+t} d\mu \sum_y \langle \varphi_0 \varphi_y \rangle_{0,\Lambda,\mu} \langle \varphi_x \varphi_y \rangle_{0,\Lambda,\mu} \right| \end{aligned}$$

Now for sufficiently large Λ, $\mu_c(\Lambda) + \frac{1}{2}t \leqslant \mu \leqslant \mu_c(\Lambda) + \frac{3}{2}t$, and by Proposition 2.3,

$$\sum_y \langle \varphi_0 \varphi_y \rangle_{0,\Lambda,\mu} \langle \varphi_x \varphi_y \rangle_{0,\Lambda,\mu} \leqslant \text{const}''(t)$$

Thus, the first term of (2.16) is bounded by

$$|\mu_c - \mu_c(\Lambda)| \cdot \text{const}''(t) \tag{2.17}$$

and can be made arbitrarily small (for fixed t, x) by taking Λ sufficiently large.

As for the second term of (2.16), we can directly use Lemma 2.4, and conclude that this can also be made arbitrarily small by taking Λ large. ∎

Corollary 2.6. Bounds on finite-volume expectations (2.6)–(2.9) hold also for those on infinite systems, with $\mu_c(\Lambda_0)$ replaced by μ_c where μ_c is defined by (2.10).

2.3. The Bound on the Susceptibility

Now that we have Corollary 2.6, it is easy to get the bound on the susceptibility in the infinite-volume limit. By the definition of block spin,

$$\varphi_x^n \equiv z_n^{-1/2} L^{-3n} \sum_{y \in B^n(x)} \varphi_y, \qquad z_n \equiv \prod_{k=0}^{n-1} \zeta_k \tag{2.18}$$

$$\chi_0(t) = z_n L^{2n} \chi_n(t) \tag{2.19}$$

(Because the summations in the definition of χ_n are absolutely convergent, we can interchange the order of the summation.) Here, $1/2 \leqslant z_n \leqslant 2$ (uniform in n), because $|\zeta_k - 1| \leqslant (n_0 + k)^{-3/2}$. So, if we combine the bound on L^{2n}, given by the Corollary 2.4 of I, and also use the bound on n_1 [I, Theorem 2.3(i)], we arrive at the bound (1.7) of Theorem 1.1.

2.4. The Bound on the Correlation Length

To treat the correlation length, we have to be more careful.

First, for $\langle \varphi_0 \varphi_x \rangle_{0,\mathbf{Z}^4,\mu}$ and for $\langle \varphi_0 \varphi_x \rangle_{n,\mathbf{Z}^4,\mu}$, we have the following *spectral representations*:

Proposition 2.7:

$$\frac{\langle \varphi_0 \varphi_x \rangle_{0,\mathbf{Z}^4,\mu}}{\langle \varphi_0 \varphi_0 \rangle_{0,\mathbf{Z}^4,\mu}} = \int d\rho_0(s, q)\, s^{|x_1|} \exp\left(-i \sum_2^4 q_\nu x_\nu\right) \tag{2.20}$$

Here $d\rho_0$ is a normalized measure whose support is in $[0, 1] \times [-\pi, \pi)^3$, and $\sup_s\{\text{support } d\rho_0\} = e^{-m}$, $m \equiv \xi^{-1}$. Moreover, for $x_1 \geqslant 1$,

$$\frac{\langle \varphi_0^n \varphi_x^n \rangle_{n,\mathbf{Z}^4,\mu}}{\langle \varphi_0^n \varphi_1^n \rangle_{n,\mathbf{Z}^4,\mu}} = \int d\rho_n(s, q)\, s^{x_1 - 1} \exp\left(i \sum_2^4 q_\nu x_\nu\right) \tag{2.21}$$

Here $d\rho_n$ is a normalized measure, whose support is in $[0, 1] \times [-\pi, \pi)^3$, and $\sup_s\{\text{support } d\rho_n\} = e^{-m_n}$, $m_n \equiv L^n \xi^{-1}$.

Proof of Theorem 1.2(ii), Assuming Proposition 2.7. Comparing (2.20) with the bound (2.6), we can immediately see that

$$\exp(-m_n) \leqslant \exp(-\alpha/12) \tag{2.22}$$

because if m_n was smaller than $\alpha/12$, the spectral representation would yield the two-point function decaying much more slowly than (2.6).

On the other hand, combining (2.21) with (2.9), we can get the upper bound on m_n:

$$m_n \leqslant (2L^2M)^{1/2} + \ln(500L^2M) \tag{2.23}$$

The above two bounds, together with the relation between m_0 and m_n, yield the desired bound (1.8) on $\xi(t)$. ∎

Proof of Proposition 2.7. The spectral representation for $\langle \varphi_0 \varphi_x \rangle_{0,\mathbf{Z}^4,\mu}$ is a consequence of reflection positivity with respect to bond planes and site planes (see, e.g., Ref. 6).

The representation for $\langle \varphi_0 \varphi_x \rangle_{n,\mathbf{Z}^4,\mu}$ is derived by explicitly constructing $\langle \varphi_0 \varphi_x \rangle_{n,\mathbf{Z}^4,\mu}$ from $\langle \varphi_0 \varphi_x \rangle_{0,\mathbf{Z}^4,\mu}$. That is, for $x_1 \geqslant 1$,

$$\begin{aligned}
&\langle \varphi_0^n \varphi_x^n \rangle_{n,\mathbf{Z}^4,\mu} \\
&\quad \equiv z_n \sum_{y \in B^n(0), z \in B^n(x)} \langle \varphi_y \varphi_z \rangle_{0,\mathbf{Z}^4,\mu} \\
&\quad = z_n \langle \varphi_0^2 \rangle_{0,\mathbf{Z}^4,\mu} \int d\rho_0(s, q)\, s^{L^n x_1} \sum_{|u|,|v| < L^n/2} s^{u-v} \\
&\qquad \times \exp\left(-iL^n \sum_2^4 q_\nu x_\nu \right) \\
&\qquad \times \sum_{|u_\nu|,|v_\nu| < L^n/2} \exp\left[-i \sum_2^4 (u_\nu - v_\nu)\, q_\nu \right] \\
&\quad = z_n \langle \varphi_0^2 \rangle_{0,\mathbf{Z}^4,\mu} \int d\rho_0(s, q) \\
&\qquad \times s^{L^n x_1} \left(\frac{s^{L^n/2} - s^{-L^n/2}}{s^{1/2} - s^{-1/2}} \right)^2 \\
&\qquad \times \exp\left(-iL^n \sum_2^4 q_\nu x_\nu \right) \\
&\qquad \times \prod_2^4 \left| \frac{\exp(-iL^n q_\nu/2) - \exp(iL^n q_\nu/2)}{\exp(-iq_\nu/2) - \exp(iq_\nu/2)} \right|^2
\end{aligned}$$

Defining new variables

$$s' \equiv s^{L^n}$$

$$q'_\nu \equiv L^n q_\nu \pmod{2\pi}$$

and dividing both sides by $\langle \varphi_0^n \varphi_1^n \rangle_{n,\mathbf{Z}^4,\mu}$, we can get the representation. (Because n is finite and since we know that

$$0 < \langle \varphi_0^n \varphi_1^n \rangle_{n,\mathbf{Z}^4,\mu} < \infty$$

the supremum of the support of $d\rho_n$ is e^{-m_n}.) ∎

2.5. Bound on the Renormalized Coupling

First note that g_{ren} is renormalization group-invariant. That is, for any n,

$$g_{\text{ren}} \equiv -\bar{u}_4/\chi^2 \xi^d = -\bar{u}_{4,n}/(\chi_n)^2 (\xi_n)^d$$

where

$$\xi_n \equiv (m_n)^{-1} \equiv L^{-n} \xi$$

Now taking $n \equiv n_1 + n_2$, and making use of the bounds on χ_n, $\bar{u}_{4,n}$ (Proposition 2.3), and on ξ_n [(2.22) and (2.23)], we obtain the result.

3. UPPER BOUNDS ON TWO-POINT FUNCTIONS

The upper bounds of Proposition 2.3 are provided by the following proposition (of standard form), which is derived by the standard technique of *cluster expansion*. (For a review of the cluster expansion technique see, e.g., Ref. 9.) Because our Hamiltonian $\mathscr{H}^n$ is rather complicated, we will describe our method in some detail.

Proposition 3.1. Under the assumption of Proposition 2.3, consider

$$Z(H) \equiv \int \left(\prod_{x \in \Lambda_n} d\varphi_x \right) \exp[-\mathscr{H}_n(\Phi) + (\mathbf{\Phi}, \mathbf{H})] \tag{3.1}$$

where

$$\begin{gathered} \mathbf{H} \equiv \{H_x\}_{x \in \Lambda_n}, \qquad (\mathbf{\Phi}, \mathbf{H}) \equiv \sum_{x \in \Lambda_n} \varphi_x H_x \\ H_x \in C, \qquad |H_x| \leqslant L^{-4N_0 - 1} \equiv h \end{gathered} \tag{3.2}$$

and

$$F(\mathbf{H}) \equiv \ln[Z(\mathbf{H})/Z(0)] \tag{3.3}$$

Then, F has a representation

$$F(\mathbf{H}) = \sum_{Y \subset \Lambda_n} f_Y(\mathbf{H}) \tag{3.4}$$

Here Y runs over all the paved sets in Λ_n, and $f_Y(\mathbf{H})$ depends only on $\mathbf{H}|_Y$. Moreover, as long as H_x satisfies (3.2), f_Y satisfies:

$$|f_Y(\mathbf{H})| \leqslant \exp[-4 - \tfrac{1}{8}\alpha \mathscr{L}(Y)] \tag{3.5}$$

where $\mathscr{L}(Y)$ is the length of the shortest tree on the centers of Δ's building Y (see I or Ref. 10).

Proof of (2.6) and (2.7), Assuming Proposition 3.1. Since we are considering a finite system ($\langle \varphi \rangle_n = 0$),

$$\langle \varphi_0 \varphi_x \rangle_n = \frac{\partial}{\partial H_0} \frac{\partial}{\partial H_x} F(\mathbf{H}) \Big|_{\mathbf{H}=0} \tag{3.6}$$

By the representation (3.4) and the properties of f_Y,

$$\langle \varphi_0 \varphi_x \rangle_n = \sum_{\substack{Y \subset \Lambda_n \\ Y \ni 0, x}} \frac{\partial}{\partial H_0} \frac{\partial}{\partial H_x} f_Y(\mathbf{H}) \Big|_{\mathbf{H}=0} \tag{3.7}$$

Now we estimate the derivative by the Cauchy formula:

$$\begin{aligned}
&\left| \frac{\partial}{\partial H_0} \frac{\partial}{\partial H_x} f_Y(\mathbf{H}) \Big|_{\mathbf{H}=0} \right| \\
&\quad = \left| \int_{|z|=h} \frac{dz}{2\pi i} \int_{|z'|=h} \frac{dz'}{2\pi i} \frac{1}{Z^2} \frac{1}{Z'^2} f_Y(H_0 = z, H_x = Z') \right| \\
&\quad \leqslant h^{-2} \max_{|H_0| = |H_x| = h} |f_Y| \leqslant L^{8N_0+2} \exp\left[-4 - \frac{\alpha}{8} \mathscr{L}(Y) \right]
\end{aligned}$$

Substituting this into (3.7) and estimating the sum [use (# of $Y \ni 0$, s.t. $\mathscr{L}(Y) = L^{N_0} l) \leqslant 2(8d^2)^l$ and $\mathscr{L}(Y) \geqslant |Y| - dL^{N_0}$],[10,11] we obtain

$$(3.7) \leqslant 4L^{8N_0+2} \exp[-4 - \tfrac{1}{12}\alpha(|x| - dL^{N_0})]$$

This is nothing but (2.6).

The second inequality of (2.7) can be obtained by noting that

$$\begin{aligned}
\chi_n \equiv \sum_x \langle \varphi_0 \varphi_x \rangle &= L^{-4N_0} |\Lambda_n|^{-1} \sum_{x,y} \frac{\partial}{\partial H_0} \frac{\partial}{\partial H_x} F(\mathbf{H}) \Big|_{\mathbf{H}=0} \\
&= L^{-4N_0} |\Lambda_n|^{-1} \frac{\partial^2}{\partial H^2} F(\{H_x \equiv H\}) \Big|_{H=0}
\end{aligned}$$

and estimating the right-hand side as before. ∎

Proposition 3.1 itself is proven step by step in Sections 3.1–3.3.

3.1. Decoupling Expansion

Recall I, Theorem 2.2(ii). The term $\exp(-\mathscr{H}^n)$ was given by

$$\exp(-\mathscr{H}^n) = \exp[-\tfrac{1}{2}(\Phi, (G_n^{(\mu_n)})^{-1}\Phi) - V_2(\Phi)] \times \sum_{\{X_i\}} \prod_i g_{X_i}^{nD}(\Phi) \exp[-V_{\sim D}^{\{X_i\}}(\Psi)] \tag{3.8}$$

To generate a decoupling expansion, we proceed as we did in the iteration of BST.[10,1]

1. Localize the regions where $|\varphi|$ is large (cf. Ref. 10, p. 214). Define

$$\begin{aligned} \chi_{\bar{p}}(\Phi) &\equiv \prod_{x\in\Lambda_n} \chi_{[(n_0+n)^{1/12} p_x \leqslant |\varphi_x| \leqslant (n_0+n)^{1/12}(p_x+1)]} \\ R(\bar{p}) &\equiv \bigcup_{x\in\Lambda_n} \{\Delta \subset L^{-n}\Lambda_n \mid d(\Delta, x) < (10/\alpha)\ln(p_x+1)\} \end{aligned} \tag{3.9}$$

and insert $1 = \sum_p \chi_{\bar{p}}(\Phi)$ into the integral in the definition of $Z(\mathbf{H})$.

2. Mayer-expand $\tilde{V}_{2Y}$ and $\tilde{V}_{\geqslant 4Y}$ in V_2 and $V_{\sim D}$. As a result,

$$\begin{aligned} Z(\mathbf{H}) = \sum_{\bar{p}} \sum_{\{X_i\}} \sum_{\{Y_\alpha\}} \sum_{\{Y_\beta\}} \int \prod_i g_{X_i}^{nR}(\Psi) \exp[(\Phi, \mathbf{H})|_X] \\ \times \exp\left[-\frac{a}{2}(\Phi, C+1, \Phi) + \frac{\lambda}{4}\int dx\, \mathscr{G}_{xx}\psi_x^2 \right. \\ \left. - \sum_i \sum_{Y\subset X_i} \tilde{V}_{2Y} - \frac{\lambda}{4!}\int dx\, \psi_x^4 \right] \\ \times \prod_\alpha [\exp(-\tilde{V}_{2Y_\alpha}) - 1] \prod_\beta [\exp(-\tilde{V}_{\geqslant 4Y_\beta}) - 1]\, \chi_{\bar{p}}(\Phi)\, d\Phi \end{aligned}$$

where we have defined

$$a \equiv (G_n^{(\tilde{\mu}_n)})_{00}^{-1}, \qquad a\cdot C_{xy} \equiv (G_n^{(\tilde{\mu}_n)})_{xy}^{-1} - a\cdot\delta_{xy} \tag{3.10}$$

3. Now decouple the nonlocality due to the kernels $\mathscr{A}$ (relating Ψ and Φ) and G^{-1} (or C). Let $\{U_k\}$ be the partition of the volume $L^{-n}\Lambda$ into unions of blocks $\Delta \subset L^{-n}\Lambda$ connected with respect to X_i, Y_α, Y_β, and nearest neighbor (Δ_1, Δ_2) $[\Delta_1 \subset R(p)]$. In general, for any matrix A on $L^{-n}\Lambda$ (or on Λ_n), we define A^s as

$$(A^s)_{xy} \equiv \begin{cases} A_{xy}, & x, y \in \text{same } U_k \\ s_{kk'} A_{xy}, & x\in U_k, \quad y \in U_{k'} \end{cases} \tag{3.11}$$

where $0 \leqslant s_{kk'} = s_{k'k} \leqslant 1$. Doing this for C and $\mathscr{A}$, we get

$$\tilde{Z}(\mathbf{H}) \equiv Z(\mathbf{H}) \Big/ \int d\Phi \exp[-\tfrac{1}{2}a(\Phi, \Phi)]$$

$$= \sum_{\{X_\gamma\}} \prod_\gamma \rho_{X_\gamma}(\mathbf{H}) \tag{3.12a}$$

Here $\{X_\gamma\}$ is a partition of Λ_n (i.e., $\bigcup X_\gamma = \Lambda_n$, $X_\gamma \cap X_{\gamma'} = \varnothing$ for $\gamma \neq \gamma'$), and

$$\rho_X(H) \equiv \sum_{\bar{p}} \sum_{\{X_i\}} \sum_{\{Y_\alpha\}} \sum_{\{Y_\beta\}} \int S(\mathscr{U}) \left[\prod_i g_{X_i}^{nR}(\Psi^s) \right] \exp[(\Phi, \mathbf{H})|_X]$$

$$\times \exp\left[-\frac{a}{2}(\Phi, C^s, \Phi)\Big|_X + \frac{\lambda}{4} \int_X dx\, \mathscr{G}_{xx} \psi_x^{s2} \right.$$

$$\left. - \sum_i \sum_{Y \subset X_i} \tilde{V}_{2Y} - \frac{\lambda}{4!} \int_{X \setminus R} dx\, \psi_x^{s4} \right]$$

$$\times \prod_\alpha [\exp(-\tilde{V}_{2Y_\alpha}) - 1] \prod_\beta [\exp(-\tilde{V}_{\geqslant 4 Y_\beta}) - 1]$$

$$\times \chi_{\bar{p}}(\Phi)\, d\mu_{a^{-1}}(\Phi) \tag{3.12b}$$

The summations are taken as in Ref. 10, (5.25).

4. We separate the contribution from $X = \Delta$, and turn (3.12b) into a system of disjoint polymers:

$$\tilde{Z}(\mathbf{H}) = \prod_{\Delta \subset \Lambda_n} \rho_\Delta \sum_{\{X_\gamma\}} \prod_\gamma \tilde{\rho}_{X_\gamma}(\mathbf{H}) \tag{3.13}$$

where

$$\tilde{\rho}_X \equiv \rho_X \Big/ \prod_{\Delta \subset X} \rho_\Delta \tag{3.13'}$$

and the summation runs over $\{X_\gamma\}$, $X_\gamma \cap X_{\gamma'} = \varnothing$ for $\gamma \neq \gamma'$.

5. Now we can take the logarithms by a standard method (see, e.g., Refs. 9, 12, and 13). Define

$$\mathscr{f}_Y \equiv \begin{cases} \ln \rho_\Delta, & Y = \Delta \\ \displaystyle\sum_{m=1}^\infty \sum_{\substack{(X_1, X_2, ..., X_m) \\ \bigcup X_i = Y}} \frac{1}{m!} \left(\prod_i \rho_{X_i} \right) a(X_1, X_2, ..., X_m), & Y \neq \Delta \end{cases} \tag{3.14}$$

$$a(X_1, X_2, ..., X_m) \equiv \sum_G \prod_{ij} (U_{ij} - 1)$$

$$U_{ij} \equiv \begin{cases} 1 & \text{for} \quad X_i \cap X_j = \varnothing \\ 0 & \text{otherwise} \end{cases}$$

where G runs over all the connected graphs on $(1, 2,..., m)$. Then we have

$$\ln \tilde{Z}(\mathbf{H}) = \sum_{Y \subset \Lambda_n} \tilde{f}_Y(\mathbf{H}) \tag{3.15}$$

and

$$F(H) = \sum_{Y \subset \Lambda_n} f_Y(\mathbf{H}) \tag{3.16}$$

with

$$f_Y(\mathbf{H}) \equiv \tilde{f}_Y(\mathbf{H}) - \tilde{f}_Y(0) \tag{3.16$'$}$$

By the construction, f_Y depends only on $\mathbf{H}|_Y$.

This is the desired decoupling expansion.

3.2. Estimation of Polymer Activities

First, let us estimate ρ_Δ of (3.12b):

$$\begin{aligned}\rho_\Delta(\mathbf{H}) \equiv & \sum_{\{Y_\alpha\} = \varnothing, \{\Delta\}} \sum_{\{Y_\beta\} = \varnothing, \{\Delta\}} \int \exp[(\Phi, H)|_\Delta] \\ & \times \exp\left[-\frac{a}{2} (\Phi, C^0, \Phi)\Big|_\Delta + \frac{\lambda}{4} \int_\Delta dx\, \mathscr{G}_{xx} (\psi_x^0)^2 - \frac{\lambda}{4!} \int_\Delta dx\, (\psi_x^0)^4 \right] \\ & \times \prod_\alpha [\exp(-\tilde{V}_{2Y_\alpha}) - 1] \prod_\beta [\exp(-\tilde{V}_{\geqslant 4Y_\beta}) - 1] \\ & \times \chi_0(\Phi)\, d\mu_{a^{-1}}(\Phi)\end{aligned} \tag{3.12c}$$

By the inductive assumptions,

$$\begin{aligned} |\exp(-\tilde{V}_{2\Delta}) - 1| &\leqslant (n_0 + n)^{-1/2} \\ |\exp(-\tilde{V}_{\geqslant 4\Delta}) - 1| &\leqslant (n_0 + n)^{-1/3} \\ \left| \frac{\lambda}{4} \int_\Delta dx\, \mathscr{G}_{xx} (\psi_x^0)^2 - \frac{\lambda}{4!} \int_\Delta dx\, (\psi_x^0)^4 \right| &\leqslant (n_0 + n_2)^{-1/3} \sum_{x \in \Delta} (\varphi_x)^2 \\ \left| \sum_{x \in \Delta} \varphi_x H_x \right| &\leqslant \tfrac{1}{4} \sum_{x \in \Delta} |H_x| + \sum_{x \in \Delta} \varphi_x |H_x| \\ &\leqslant (4L)^{-1} + L^{-4N_0 - 1} \sum_{x \in \Delta} \varphi_x^2 \end{aligned} \tag{3.17}$$

Moreover, using (A.3), we obtain

$$|\tfrac{1}{2} a(\Phi_\Delta, C^0 \Phi_\Delta)| \leqslant \tfrac{1}{2} a M^{-1/2} 3^9 \sum_{x \in \Delta} \varphi_x^2 \tag{3.18}$$

Substituting these into (3.12c) and estimating the integral, we immediately obtain

$$\exp(-3/L) \leqslant \rho_\Delta(H) \leqslant \exp(3/L) \tag{3.19}$$

Now let us turn to the estimation of ρ_Y, $Y > \Delta$ [see (3.12b)]. We allow complex values for $s_{kk'}$

$$|s_{kk'}| \leqslant 2r'' \exp[\alpha d(U_k, U_{k'})] \tag{3.20}$$

with

$$r'' = \exp(12 + 4\alpha L^{N_0}) \leqslant r \tag{3.20'}$$

For these $s_{kk'}$, and for Φ in the support of $\chi_{\bar{p}}$,

$$\psi^s \in \mathscr{D}^{(\bar{\mu}_n)}(R(p), X)$$

So we can use all the inductive assumptions D_n, E_n of I and can use bounds similar to (3.17). For example,

$$|\exp(-\tilde{V}_{2Y}) - 1| \leqslant (n_0 + n)^{-1/2} \exp[-\alpha \mathscr{L}(Y)]$$

We first estimate the contribution from the $\bar{p} = 0$ term. Since the coefficient of $\sum \varphi^2$ is very small, we have

$$\int d\mu_a(\Phi)(\cdots) \leqslant (1 + \tfrac{1}{2} L^{-4N_0})^{L^{4N_0}|X|} \leqslant \exp(|X|)$$

Using the Cauchy estimate to bound the s-derivatives [recall (3.20)]

$$\partial_s \to r''^{-1} \exp[-\alpha d(U_k, U_{k'})] \tag{3.21}$$

and evaluating the summation over Y_α, Y_β, we finally obtain

$$\begin{aligned} &|\text{contribution from } \bar{p} = 0 \text{ term to } \rho_x| \\ &\quad \leqslant \exp(-6 - \tfrac{4}{5}\alpha \mathscr{L}(x) + 4\,|X|) \end{aligned} \tag{3.22}$$

The contribution from the $\bar{p} \neq 0$ term is bounded similarly. This time, using the exponentially small factor $\exp[-(n_0 + n)^{1/6} \sum p_x^2]$ coming from $\chi_{\bar{p}}$, we have

$$\begin{aligned} &|\text{contribution from } \bar{p} \neq 0 \text{ term to } \rho_x| \\ &\quad \leqslant \exp[-(n_0 + n)^{1/6}] \exp[-6 - \tfrac{1}{2}\alpha \mathscr{L}(x) + 4\,|X|] \end{aligned} \tag{3.22'}$$

Combining (3.22) and (3.22′), we obtain

$$|\rho_X| \leqslant (5/4) \exp[-6 - \tfrac{1}{2}\alpha \mathscr{L}(X) + 4\,|X|] \tag{3.23}$$

If we use the estimate on ρ_Δ, (3.19), we find

$$|\tilde{\rho}_X| \leqslant (5/4)\exp[-6-\tfrac{1}{2}\alpha\mathscr{L}(X)+5\,|X|\,] \\ \leqslant (3/2)\exp[-6-\tfrac{1}{4}\alpha\mathscr{L}(X)] \tag{3.24}$$

In deriving the second inequality, we used

$$\mathscr{L}(X) \geqslant (|X|-1)\,L^{N_0} \geqslant |X|\,L^{N_0}/2 \qquad \text{for} \quad |X|>1 \tag{3.25}$$

3.3. Taking the Logarithm

Now that we have proven the convergence of the expansion (3.13′), we can take the logarithm by the standard formula (3.14). The result is

$$|\tilde{f}_\Delta| \leqslant 4/L \leqslant \tfrac{1}{2}\exp(-4)$$
$$|\tilde{f}_Y| \leqslant \exp[-5-\tfrac{1}{8}\alpha\mathscr{L}(Y)]$$

Thus, f_Y of (3.16) obeys the bound (3.5).

4. LOWER BOUNDS ON TWO-POINT FUNCTIONS

In general, it seems quite difficult to derive good *lower bounds* on correlations (especially on those with massive decay) simply from the cluster expansion.[5] We here derive the desired lower bounds by comparing the expectation $\langle\cdots\rangle_n$ with the Gaussian one $\langle\cdots\rangle_G$:

$$\langle\cdots\rangle_G \equiv \int d\mu_{G_n^{(\bar{\mu}_n)}}(\cdots) \tag{4.1}$$

where $d\mu_G$ is a normalized Gaussian measure with mean zero, covariance G.

Proposition 4.1. Under the assumption of Proposition 2.3,

$$|\langle\varphi_0^n\varphi_x^n\rangle_n - \langle\varphi_0^n\varphi_x^n\rangle_G| \leqslant \tfrac{1}{15}L^{8N_0+2}(n_0+n)^{-1/2} \tag{4.2}$$

[Proof of (2.7) and (2.9), assuming Proposition 4.1.]

[5] One can sometimes obtain both lower and upper bounds on $\langle\cdots\rangle$ by performing a "cluster expansion for $\ln\langle\cdots\rangle$," as was done in Ref. 14. But such bounds are not sufficiently sharp for our purposes (especially for $\langle\varphi_0\varphi_2\rangle$).

By I, Proposition A.4, (A.7), we have a lower bound on the Gaussian propagator:

$$\langle(\varphi_0^n)^2\rangle_G = (G)_{00} \geqslant \mu_n^{-1}(1 - 2\pi/\sqrt{\mu_n})$$
$$\geqslant (2L^2M)^{-1}(1 - 2\pi/\sqrt{M}) \tag{4.3}$$

As $n_0 \geqslant \bar{n}_0 \gg M^2$, (4.2) and (4.3) give

$$\langle(\varphi_0^n)^2\rangle_n \geqslant (2L^2M)^{-1}(1 - 2\pi/\sqrt{M}) - \tfrac{1}{15}L^{8N_0+2}(n_0+n)^{-1/2}$$
$$\geqslant (3L^2M)^{-1} \tag{4.4}$$

By the Griffiths I inequality,[8] $\langle\varphi_0^n\varphi_x^n\rangle_n \geqslant 0$, and thus

$$\chi_n \equiv \sum_x \langle\varphi_0^n\varphi_x^n\rangle_n \geqslant \langle(\varphi_0^n)^2\rangle_n \geqslant (3L^2M)^{-1}$$

This proves (2.7).

To prove (2.9), we combine (4.2) and I, Proposition A.4, (A.8), for t sufficiently small {i.e., n sufficiently large that $L^{8N_0}(n_0+n)^{-1/2} \ll \exp[-(L^2M)^{1/2}]$}. ∎

Proof of Proposition 4.1. Since this is a statement about an *upper bound* on the difference of correlations, we expect that we can prove it by a suitable *cluster expansion* technique.

We first express the difference $\langle\cdots\rangle_n - \langle\cdots\rangle_G$ in a more tractable form.

Definition 4.2. Define $(e^{-\mathscr{H}^n})_t$ $(0 \leqslant t \leqslant 1)$ as

$$(e^{-\mathscr{H}^n})_t \equiv \exp[-\tfrac{1}{2}(\Phi, G^{-1}, \Phi) - t \cdot V_2]$$
$$\times \sum_{\{X_i\}} \prod_i (tg_{X_i}^{nR}) \exp(-tV_{\sim D}) \tag{4.5}$$

and define

$$Z(\mathbf{H})_t \equiv \int d\Phi\, (e^{-\mathscr{H}^n})_t\, e^{(\mathbf{H}, \mathbf{\Phi})} \tag{4.6}$$

We use the subscript G to denote the corresponding Gaussian quantity:

$$Z(\mathbf{H})_G \equiv \int d\Phi \exp[-\tfrac{1}{2}(\Phi, G^{-1}\Phi) + (\mathbf{H}, \mathbf{\Phi})] \tag{4.6$'$}$$

As in Section 3, we also define (asterisk subscript denotes t or G)

$$\tilde{Z}(\mathbf{H})_* \equiv Z(\mathbf{H})_* \Big/ \int d\Phi \exp[-\tfrac{1}{2}a(\Phi, \Phi)] \tag{4.7}$$

By the definition, we have

$$\begin{aligned}&\langle\varphi_0\varphi_x\rangle_n-\langle\varphi_0\varphi_x\rangle_G\\&=\frac{\partial}{\partial H_0}\frac{\partial}{\partial H_x}[\ln\tilde{Z}(\mathbf{H})_{t=1}-\ln\tilde{Z}(\mathbf{H})_{t=0}\\&\quad+\ln\tilde{Z}(\mathbf{H})_{t=0}-\ln\tilde{Z}(\mathbf{H})_G]\Big|_{H=0}\end{aligned}\tag{4.8}$$

To estimate these terms, we use the following two lemmas.

Lemmas 4.3. For $H_x\in\mathbf{C}$, $|H_x|\leqslant L^{-4N_0-1}$, we can write (asterisk subscript stands for $t=0$ or G)

$$\ln\tilde{Z}(\mathbf{H})_*=\sum_{Y\subset\Lambda_n}\tilde{f}_Y(\mathbf{H})_*\tag{4.9}$$

Moreover,

$$|\tilde{f}_Y(\mathbf{H})_G-\tilde{f}_Y(\mathbf{H})_{t=0}|\leqslant\exp[-(n_0+n)^{1/6}-4-\tfrac{1}{8}\alpha\mathscr{L}(Y)]\tag{4.10}$$

Lemma 4.4. For $|H_x|\leqslant L^{-4N_0-1}$ and $|t|\leqslant(n_0+n)^{1/2}+1$, we have

$$\ln\tilde{Z}(\mathbf{H})_t=\sum_{Y\subset\Lambda_n}\tilde{f}_Y(\mathbf{H})_t\tag{4.11}$$

and

$$|\tilde{f}_Y(\mathbf{H})_t|\leqslant\exp[-4-\tfrac{1}{8}\alpha\mathscr{L}(Y)]\tag{4.12}$$

Here $\tilde{f}_Y(\mathbf{H})_t$ depends only on $\mathbf{H}|_Y$.

These lemmas are proven at the end of this section.

Now by Lemma 4.3 and by the Cauchy estimate in $\mathbf{H}$, the third and the fourth terms of (4.8) are bounded as

$$\begin{aligned}&\left|\frac{\partial}{\partial H_0}\frac{\partial}{\partial H_x}[\ln\tilde{Z}(\mathbf{H})_G-\ln\tilde{Z}(\mathbf{H})_{t=0}]\Big|_{\mathbf{H}=0}\right|\\&\leqslant\sum_{Y\ni0,x}\left|\left.\frac{\partial}{\partial H_0}\frac{\partial}{\partial H_x}\right|_{\mathbf{H}=0}[\tilde{f}_Y(\mathbf{H})_G-\tilde{f}_Y(\mathbf{H})_{t=0}]\right|\\&\leqslant L^{8N_0+2}\exp[-(n_0+n)^{1/2}]\sum_{Y\ni0,x}\exp[-4-\tfrac{1}{8}\alpha\mathscr{L}(Y)]\\&\leqslant L^{8N_0+2}\exp[-3-(n_0+n)^{1/2}]\end{aligned}\tag{4.13}$$

On the other hand, expressing the first and second terms of (4.8) as

$$\frac{\partial}{\partial H_0}\frac{\partial}{\partial H_x}[\ln \tilde{Z}(\mathbf{H})_{t=1} - \ln \tilde{Z}(\mathbf{H})_{t=0}]$$

$$= \int_0^1 dt \left.\frac{d}{dt}\frac{\partial}{\partial H_0}\frac{\partial}{\partial H_x}\right|_{\mathbf{H}=0} \sum_{Y \ni 0,x} \tilde{f}_Y(\mathbf{H})_t \tag{4.14}$$

and using the Cauchy estimates *both in H and t* (and Lemma 4.4) to bound H- and t-derivatives,

$$(4.14) \leqslant (n_0+n)^{-1/2} L^{8N_0+2} e^{-3} \tag{4.15}$$

we obtain from (4.9), (4.13), and (4.15) the desired bound. ∎

Proof of Lemma 4.3. Recall, by the definition of $Z(\mathbf{H})_t$, that

$$Z(\mathbf{H})_{t=0} = \int d\Phi\, \chi_0(\Phi) \exp[-\tfrac{1}{2}(\Phi, G^{-1}, \Phi) + (\mathbf{H}, \mathbf{\Phi})]$$

while

$$Z(\mathbf{H})_{\mathrm{G}} = \int d\Phi \exp[-\tfrac{1}{2}(\Phi, G^{-1}, \Phi) + (\mathbf{H}, \mathbf{\Phi})]$$

We perform the cluster expansion for both of them. But now, we have to take only two steps:

1. Localize the "large-Φ" regions by $\chi_{\bar{p}}(\Phi)$ of (3.9).
2. Decouple the nonlocality caused by G^{-1}, using $(G^{-1})_s$, as in Section 3.

We proceed in the same way as in Section 3 (in fact in a much easier way). By the definition, the contribution from the $\bar{p}=0$ term is common to $Z_{t=0}$ and Z_{G}, and the difference lies only in $\bar{p} \neq 0$ terms in Z_{G}. We thus have (asterisk subscript denotes 0 or G)

$$\tilde{Z}(\mathbf{H})_* = \left[\prod_{\Delta \subset \Lambda_n} (\rho_\Delta)_*\right] \sum_{\{X_\gamma\}} \prod_\gamma (\tilde{\rho}_X)_*$$

where

$$(\rho_\Delta)_0 = (\rho_\Delta)_{\mathrm{G}} \quad \text{(contribution from } \bar{p}=0\text{)}$$

$$|(\bar{p}=0 \text{ in } \tilde{\rho}_X)_0| = |(\bar{p}=0 \text{ in } \tilde{\rho}_X)_{\mathrm{G}}| \leqslant \exp[-4 - \tfrac{1}{4}\alpha\mathscr{L}(x)]$$

$$|(\bar{p} \neq 0 \text{ in } \tilde{\rho}_X)_{\mathrm{G}}| \leqslant \exp[-(n_0+n)^{1/6} - \tfrac{1}{4}\alpha\mathscr{L}(x) + 4\,|X|]$$

Thus,

$$|(\tilde{\rho}_X)_0 - (\tilde{\rho}_X)_G| \leqslant \exp[-(n_0+n)^{1/6} - \tfrac{1}{4}\alpha\mathscr{L}(x) + 4\,|X|]$$

Now taking the logarithm by (3.14), we get the desired lemma. ∎

Proof of Lemma 4.4. This is almost same as that of Proposition 3.1. The only difference lies in the fact that we have introduced $|t| \leqslant (n_0+n)^{1/2}+1$, which is multiplied to $\tilde{V}_{2Y}$, $\tilde{V}_{\geqslant 4Y}$, etc. But now, since we are using $(n_0+n)^{1/12}$ to distinguish between large and small fields, we can refine bounds on $\tilde{V}_{2Y}$, etc. For example, since

$$|\tilde{V}_{\geqslant 4Y}| \leqslant O(1)(n_0+n)^{-1/2} \exp[-\alpha\mathscr{L}(Y)] \qquad \text{on} \quad 3\mathscr{K}(Y)$$

we have now (by the minimum–maximum principle)

$$|\tilde{V}_{\geqslant 4Y}| \leqslant O(1)(n_0+n)^{-7/6} \exp[-\alpha\mathscr{L}(Y)]$$

Also, tg_X^{nR} appears only when there is a nonempty large-field region $R \neq \varnothing$, and thus t is multiplied by $\exp[-(n_0+n)^{1/2}]$ and harmless.

In this way, the proof of Proposition 3.1 carries over to this case with minor changes, and we obtain Lemma 4.4. ∎

5. BOUND ON TRUNCATED FOUR-POINT FUNCTION

The bound on $\bar{u}_{4,n}$ is obtained in the same spirit as that of Section 4.

Proposition 5.1. Under the assumption of Proposition 2.3,

$$|\bar{u}_{4,n} - (\bar{u}_{4,n})_G| \leqslant \tfrac{1}{8} L^{16N_0+4}(n_0+n)^{-1/2} \tag{5.1}$$

where $(\bar{u}_{4,n})_G$ is $\bar{u}_4$ in the expectation $\langle\cdots\rangle_G$ [see (4.1)].

Proof of (2.8), assuming Proposition 5.1. Obvious, because $(\bar{u}_{4,n})_{\text{Gauss}} \equiv 0$. ∎

Proof of Proposition 5.1. This can be proven in almost the same way as Proposition 4.1 (estimating the difference between our theory and the Gaussian one). We omit the details. ∎

APPENDIX

We prove the following proposition.

Proposition A.1. Let $a \equiv [(G_n^{(\mu_n)})^{-1}]_{00}$, and write

$$[(G_n^{(\mu_n)})^{-1}]_{xy} \equiv a\delta_{xy} + aC_{xy} \tag{A.1}$$

Then for $\mu_n \geqslant (200 \cdot 4 \cdot 3^d)^2$,

$$\mu_n(1 - 3^{d+8}\mu_n^{-1/2}) \leqslant a \leqslant \mu_n(1 + 3^{d+8}\mu_n^{-1/2}) \tag{A.2}$$

and

$$|C_{0x}| \leqslant (3^{d+7}\mu_n^{-1/2})^{|x|_\infty} \tag{A.3}$$

Proof. We omit the subscript (μ_n). We use Neumann series to get G_n^{-1}, that is, writing

$$G_{xy} \equiv G_{00}(\delta_{xy} + \bar{G}_{xy})$$

(Note that $\bar{G}_{xy} = 0$ for $x = y$.)

$$\begin{aligned}(G^{-1})_{xy} &= (G_{00})^{-1}[(1+\bar{G})^{-1}]_{xy} \\ &= (G_{00})^{-1}\left(\delta_{xy} - \bar{G}_{xy} + \sum_z \bar{G}_{xz}\bar{G}_{zy} - \sum_{z,u} \bar{G}_{xz}\bar{G}_{zu}\bar{G}_{uy} + \cdots\right)\end{aligned}$$

Now we can use I, Proposition A.4 to bound the summations. For example,

$$\begin{aligned}&\left|\sum_{z_1, z_2, \ldots, z_m} \bar{G}_{xz_1}\bar{G}_{z_1z_2}\bar{G}_{z_2z_3}\cdots\bar{G}_{z_my}\right| \\ &\leqslant \sum_{z_1, z_2, \ldots, z_m, (z_i \neq z_{i+1})} (200\mu_n^{-1/2})^{|x-z_1|_\infty + |z_1-z_2|_\infty + \cdots + |z_m - y|_\infty} \\ &\leqslant (200 \cdot 4 \cdot 3^d\mu_n^{-1/2})^{|x-y|_\infty}\left[\sum_{z\neq 0} (4^{-1} \cdot 3^{-d})^{|z|_\infty}\right]^m \\ &\leqslant (200 \cdot 4 \cdot 3^d\mu_n^{-1/2})^{|x-y|_\infty} 3^{-m}\end{aligned}$$

In the above we used

$$\sum_{z\neq 0} (4^{-1} \cdot 3^{-d})^{|z|_\infty} \leqslant \sum_{l=0}^{\infty} (4^{-1} \cdot 3^{-d})^l(2l+1)^d$$

We thus have (for $x \neq y$)

$$\begin{aligned}\frac{(G^{-1})_{xy}}{(G_{00})^{-1}} &\leqslant (200 \cdot 4 \cdot 3^d\mu_n^{-1/2})^{|x-y|_\infty} \sum_{m=1}^{\infty} 3^{-m} \\ &\leqslant \tfrac{1}{2}(200 \cdot 4 \cdot 3^d\mu_n^{-1/2})^{|x-y|_\infty}\end{aligned}$$

$x = y$ can be treated similarly. ∎

ACKNOWLEDGMENTS

We would like to thank Prof. M. Kato, Prof. M. Suzuki, Dr. K. R. Ito, Dr. T. Hattori, and Dr. K. I. Kondo for valuable discussions and encouragement.

NOTE ADDED IN PROOF

Though not used in the paper, Proposition 4.1 can be improved as:

$$\text{L.H.S. of } (4.2) \leqslant O((n_0+n)^{-1/2}) \exp(-\tfrac{1}{12}\alpha\,|x|)$$

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